On the derivative of the associated Legendre function of the first kind of integer degree with respect to its order

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Abstract

The derivative of the associated Legendre function of the first kind of integer degree with respect to its order, $\partial P_n^\mu(z)/\partial\mu$, is studied. After deriving and investigating general formulas for μ arbitrary complex, a detailed discussion of $[\partial P_n^\mu(z)/\partial\mu]_{\mu=\pm m}$, where m is a non-negative integer, is carried out. The results are applied to obtain several explicit expressions for the associated Legendre function of the second kind of integer degree and order, $Q_n^{\pm m}(z)$. In particular, we arrive at formulas which generalize to the case of $Q_n^{\pm m}(z)$ $(0 \leqslant m \leqslant n)$ the well-known Christoffel's representation of the Legendre function of the second kind, $Q_n(z)$. The derivatives $[\partial^2 P_n^\mu(z)/\partial\mu^2]_{\mu=m}$, $[\partial Q_n^\mu(z)/\partial\mu]_{\mu=m}$ and $[\partial Q_{-n-1}^\mu(z)/\partial\mu]_{\mu=m}$, all with $m > n$, are also evaluated.

KEY WORDS: Legendre functions; parameter derivative; special functions

AMS subject classification: 33C45, 33C05

1 Introduction

It is the purpose of the present paper to contribute to the theory of special functions of mathematical physics and chemistry. Specifically, we shall add to the knowledge about the associated Legendre functions of the first, $P_\nu^\mu(z)$, and second, $Q_\nu^\mu(z)$, kinds (cf, e.g., [1–29]). We shall touch two particular problems. First, we shall investigate the derivative of the associated Legendre function of the first kind of integer degree with respect to its order. Second, it will be shown that the results of that investigation may be used to construct, in a straightforward and unified manner, several known representations of the associated Legendre functions of the second kind of integer degree and order, $Q_n^m(z)$; in the past those representations were obtained by other authors with the use of a variety of, usually more complicated, techniques. In addition, some possibly new expressions for $Q_n^m(z)$ will be also presented.

The literature concerning the derivative $\partial P_\nu^\mu(z)/\partial\mu$ is very limited. Surprisingly, no relevant expressions have been given in [29], which otherwise contains a large collection of parameter derivatives of various special functions. Several variants of the formula

$$\left.\frac{\partial P_\nu^\mu(z)}{\partial\mu}\right|_{\mu=0} = \psi(\nu+1)P_\nu(z) + Q_\nu(z), \tag{1.1}$$

where

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta)}\frac{\mathrm{d}\Gamma(\zeta)}{\mathrm{d}\zeta} \tag{1.2}$$

is the digamma function [14, 20, 23, 25, 30], may be found in [23, page 178]. Robin [18, Eq. (333) on page 175] gave the following representation[1] of $\partial P_\nu^\mu(z)/\partial\mu$:

$$\begin{aligned}\frac{\partial P_\nu^\mu(z)}{\partial\mu} &= \frac{1}{2}P_\nu^\mu(z)\ln\frac{z+1}{z-1} \\ &\quad+\left(\frac{z+1}{z-1}\right)^{\mu/2}\sum_{k=0}^{\infty}\frac{\Gamma(\nu+k+1)\psi(k-\mu+1)}{k!\Gamma(\nu-k+1)\Gamma(k-\mu+1)}\left(\frac{z-1}{2}\right)^k \qquad (|z-1|<2).\end{aligned} \tag{1.3}$$

Brown [31], being apparently unaware of the Robin's finding, rederived the expression in Eq. (1.3) in the particular case of $\nu = n \in \mathbb{N}$ and used it to find a representation of $Q_n^m(z)$ suitable for numerical purposes. An application of the derivative $\partial P_\nu^\mu(z)/\partial\mu$ to the evaluation of the integral $\int_{\mathcal{C}} \mathrm{d}z\ [P_\nu^\mu(z)]^2/(z^2-1)$, where $\mathcal{C}$ is the contour starting from $z=0$ and returning to it after a positive circuit round the point $z=1$, was presented by Watson [32].

The detailed plan of the paper is as follows. In section 2, we shall summarize these facts about the associated Legendre functions of the first and second kinds of integer degrees which will find applications in later parts of the work. In section 3, the derivative of the associated Legendre function of the first kind, $\partial P_n^\mu(z)/\partial\mu$ (with $z \in \mathbb{C}\setminus[-1,1]$), is investigated. After general considerations forming section 3.1, where no restrictions are imposed on the order μ of the Legendre function, in sections 3.2 and 3.3 we focus on the derivatives $[\partial P_n^\mu(z)/\partial\mu]_{\mu=\pm m}$ with $m \in \mathbb{N}$. The cases of $0 \leqslant m \leqslant n$ and $m > n$ are treated separately. A brief discussion of the derivative $\partial P_n^\mu(x)/\partial\mu$ (with $x \in [-1,1]$) is contained in section 3.4. Section 4 presents some applications of the results of section 3. In section 4.1, we obtain several explicit representations of the associated Legendre function of the second kind of integer degree and order, $Q_n^{\pm m}(z)$ and $Q_n^{\pm m}(x)$. Some of these representations seem to be new, particularly these ones which generalize the well-known Christoffel's formulas for $Q_n(z)$ and $Q_n(x)$. Finally, in sections 4.2 and 4.3, the derivatives $[\partial^2 P_n^\mu(z)/\partial\mu^2]_{\mu=m}$, $[\partial Q_n^\mu(z)/\partial\mu]_{\mu=m}$, and $[\partial Q_{-n-1}^\mu(z)/\partial\mu]_{\mu=m}$, all with $m > n$, are expressed in terms of the derivatives $[\partial P_n^\mu(\pm z)/\partial\mu]_{\mu=-m}$.

Throughout the paper, we shall be adopting the notational convention according to which $\nu \in \mathbb{C}$, $\mu \in \mathbb{C}$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, $x \in [-1,1]$, and $z \in \mathbb{C}\setminus[-1,1]$, the latter with

$$-\pi < \arg(z) < \pi, \qquad -\pi < \arg(z\pm 1) < \pi, \tag{1.4}$$

hence,

$$-z = \mathrm{e}^{\mp\mathrm{i}\pi}z, \qquad -z+1 = \mathrm{e}^{\mp\mathrm{i}\pi}(z-1), \qquad -z-1 = \mathrm{e}^{\mp\mathrm{i}\pi}(z+1) \qquad (\arg(z) \gtrless 0). \tag{1.5}$$

Furthermore, it will be understood that if the upper limit of a sum is less by unity than the lower one, then the sum vanishes identically.

The definitions of the associated Legendre functions of the first and second kinds used in the paper are those of Hobson [12].

2 The associated Legendre functions of integer degrees

The material presented in this section, to be referred to in later parts of the paper, is based primarily on [12, 17, 18].

[1] The notation used in the present work differs in some respects from that adopted by Robin in [17–19]. In particular, the digamma function used by Robin was defined as

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta+1)}\frac{\mathrm{d}\Gamma(\zeta+1)}{\mathrm{d}\zeta}$$

rather than as in our Eq. (1.2). Also, one should be warned that in Eq. (333) on page 175 in [18] the factor $p!$ in the denominator of a summand is missing.

2.1 The associated Legendre function of the first kind of integer degree

The associated Legendre function of the first kind of a non-negative integer degree n and an arbitrary complex order μ may defined as

$$P_n^{\mu}(z)=(\pm)^n\frac{1}{\Gamma(1\mp\mu)}\frac{\Gamma(n\mp\mu+1)}{\Gamma(n-\mu+1)}\left(\frac{z+1}{z-1}\right)^{\mu/2}{}_2F_1\left(-n,n+1;1\mp\mu;\frac{1\mp z}{2}\right), \tag{2.1}$$

or equivalently as

$$P_n^{\mu}(z) = \frac{1}{\Gamma(1\mp\mu)}\frac{\Gamma(n\mp\mu+1)}{\Gamma(n-\mu+1)}\left(\frac{z+1}{z-1}\right)^{\mu/2}\left(\frac{z\pm1}{2}\right)^n{}_2F_1\left(-n,-n\mp\mu;1\mp\mu;\frac{z\mp1}{z\pm1}\right), \tag{2.2}$$

with either upper or lower signs chosen. Both hypergeometric functions in Eq. (2.1) are proportional to the Jacobi polynomial [23, 25] $P_n^{(-\mu,\mu)}(z)$; in terms of the latter, one has

$$P_n^{\mu}(z)=\frac{n!}{\Gamma(n-\mu+1)}\left(\frac{z+1}{z-1}\right)^{\mu/2}P_n^{(-\mu,\mu)}(z). \tag{2.3}$$

Combining this with the Rodrigues-type formula for the Jacobi polynomials, which is

$$P_n^{(\alpha,\beta)}(z)=\frac{1}{2^n n!}(z-1)^{-\alpha}(z+1)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-1)^{n+\alpha}(z+1)^{n+\beta}\right], \tag{2.4}$$

leads to the following Rodrigues-type representation of $P_n^{\mu}(z)$:

$$P_n^{\mu}(z)=\frac{1}{2^n\Gamma(n-\mu+1)}\left(\frac{z-1}{z+1}\right)^{\mu/2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-1)^{n-\mu}(z+1)^{n+\mu}\right]. \tag{2.5}$$

The function $P_n^{\mu}(z)$ is single-valued in the complex z-plane with a cut along the real axis from $-\infty$ to 1. For a negative integer degree, the associated Legendre function of the first kind is defined through the relationship

$$P_{-n-1}^{\mu}(z)=P_n^{\mu}(z). \tag{2.6}$$

For the sake of later applications, it is convenient to rewrite Eq. (2.1) as

$$P_n^{\mu}(z)=(\pm)^n\frac{\Gamma(n\mp\mu+1)}{\Gamma(n-\mu+1)}\left(\frac{z+1}{z-1}\right)^{\mu/2}\sum_{k=0}^{n}(\pm)^k\frac{(k+n)!}{k!(n-k)!\Gamma(k\mp\mu+1)}\left(\frac{z\mp1}{2}\right)^k \tag{2.7}$$

and Eq. (2.2) as

$$\begin{aligned}P_n^{\mu}(z) &= n!\Gamma(n+\mu+1)\left(\frac{z+1}{z-1}\right)^{\mu/2}\left(\frac{z\pm1}{2}\right)^n\\ &\quad\times\sum_{k=0}^{n}\frac{1}{k!(n-k)!\Gamma(k\mp\mu+1)\Gamma(n-k\pm\mu+1)}\left(\frac{z\mp1}{z\pm1}\right)^k.\end{aligned} \tag{2.8}$$

From either of Eqs. (2.1), (2.2), (2.7) or (2.8), the function $P_n^{\mu}(z)$ is seen to possess the reflection property

$$P_n^{\mu}(z)=(-)^n\frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu+1)}P_n^{-\mu}(-z). \tag{2.9}$$

If $\mu=\pm m$, then it holds that

$$P_n^{m}(z)\equiv0 \qquad (m>n), \tag{2.10}$$

$$P_n^{\pm m}(-z)=(-)^nP_n^{\pm m}(z) \qquad (0\leqslant m\leqslant n), \tag{2.11}$$

$$P_n^{-m}(z) = \frac{(n-m)!}{(n+m)!} P_n^m(z) \qquad (0 \leqslant m \leqslant n). \tag{2.12}$$

If $0 \leqslant m \leqslant n$, from Eqs. (2.7) and (2.8) we have

$$P_n^m(z) = \left(\frac{z^2-1}{4}\right)^{m/2} \sum_{k=0}^{n-m} \frac{(k+n+m)!}{k!(k+m)!(n-m-k)!} \left(\frac{z-1}{2}\right)^k \qquad (0 \leqslant m \leqslant n), \tag{2.13}$$

$$P_n^m(z) = (-)^n \frac{(n+m)!}{(n-m)!} \left(\frac{z+1}{z-1}\right)^{m/2} \sum_{k=0}^{n} (-)^k \frac{(k+n)!}{k!(k+m)!(n-k)!} \left(\frac{z+1}{2}\right)^k \qquad (0 \leqslant m \leqslant n), \tag{2.14}$$

$$P_n^m(z) = n!(n+m)! \left(\frac{z \mp 1}{z \pm 1}\right)^{m/2} \left(\frac{z \pm 1}{2}\right)^n \times \sum_{k=0}^{n-m} \frac{1}{k!(k+m)!(n-k)!(n-m-k)!} \left(\frac{z \mp 1}{z \pm 1}\right)^k \qquad (0 \leqslant m \leqslant n), \tag{2.15}$$

and also

$$P_n^{-m}(z) = \left(\frac{z-1}{z+1}\right)^{m/2} \sum_{k=0}^{n} \frac{(k+n)!}{k!(k+m)!(n-k)!} \left(\frac{z-1}{2}\right)^k \qquad (0 \leqslant m \leqslant n), \tag{2.16}$$

$$P_n^{-m}(z) = (-)^{n+m} \frac{(n-m)!}{(n+m)!} \left(\frac{z^2-1}{4}\right)^{m/2} \times \sum_{k=0}^{n-m} (-)^k \frac{(k+n+m)!}{k!(k+m)!(n-m-k)!} \left(\frac{z+1}{2}\right)^k \qquad (0 \leqslant m \leqslant n), \tag{2.17}$$

$$P_n^{-m}(z) = n!(n-m)! \left(\frac{z \mp 1}{z \pm 1}\right)^{m/2} \left(\frac{z \pm 1}{2}\right)^n \times \sum_{k=0}^{n-m} \frac{1}{k!(k+m)!(n-k)!(n-m-k)!} \left(\frac{z \mp 1}{z \pm 1}\right)^k \qquad (0 \leqslant m \leqslant n). \tag{2.18}$$

For $m > n$, the same equations yield

$$P_n^{-m}(z) = \left(\frac{z-1}{z+1}\right)^{m/2} \sum_{k=0}^{n} \frac{(k+n)!}{k!(k+m)!(n-k)!} \left(\frac{z-1}{2}\right)^k \qquad (m > n), \tag{2.19}$$

$$P_n^{-m}(z) = \frac{1}{(n+m)!(m-n-1)!} \left(\frac{z-1}{z+1}\right)^{m/2} \sum_{k=0}^{n} \frac{(k+n)!(m-k-1)!}{k!(n-k)!} \left(\frac{z+1}{2}\right)^k \qquad (m > n), \tag{2.20}$$

$$P_n^{-m}(z) = \frac{n!}{(m-n-1)!} \left(\frac{z-1}{z+1}\right)^{m/2} \left(\frac{z+1}{2}\right)^n \sum_{k=0}^{n} (-)^k \frac{(k+m-n-1)!}{k!(k+m)!(n-k)!} \left(\frac{z-1}{z+1}\right)^k \qquad (m > n), \tag{2.21}$$

and

$$
\begin{aligned}
P_n^{-m}(z) &= (-)^n \frac{n!}{(m-n-1)!} \left(\frac{z-1}{z+1}\right)^{m/2} \left(\frac{z-1}{2}\right)^n \\
&\quad \times \sum_{k=0}^{n} (-)^k \frac{(m-k-1)!}{k!(n-k)!(n+m-k)!} \left(\frac{z+1}{z-1}\right)^k \qquad (m>n). \qquad (2.22)
\end{aligned}
$$

In deriving Eqs. (2.20) to (2.22), the formula

$$
\lim_{\mu\to -m} \frac{\Gamma(n+\mu+1)}{\Gamma(n'+\mu+1)} = (-)^{n+n'} \frac{(m-n'-1)!}{(m-n-1)!} \qquad (m>n,n') \qquad (2.23)
$$

appears to be useful.

Following Hobson [12], on that part of the cut for which $z=x$ $(-1 \leqslant x \leqslant 1)$ we define the associated Legendre function of the first kind of non-negative degree as

$$
P_n^{\mu}(x) = \frac{1}{2}\left[\mathrm{e}^{\mathrm{i}\pi\mu/2} P_n^{\mu}(x+\mathrm{i}0) + \mathrm{e}^{-\mathrm{i}\pi\mu/2} P_n^{\mu}(x-\mathrm{i}0)\right] = \mathrm{e}^{\pm\mathrm{i}\pi\mu/2} P_n^{\mu}(x\pm\mathrm{i}0). \qquad (2.24)
$$

2.2 The associated Legendre function of the second kind of integer degree

The associated Legendre function of the second kind of a non-negative integer degree n and an arbitrary complex order μ may be defined by either of the two expressions:

$$
Q_n^{\mu}(z) = \frac{\pi}{2} \frac{\mathrm{e}^{\mathrm{i}\pi\mu}}{\sin(\pi\mu)} \left[P_n^{\mu}(z) - \frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu+1)} P_n^{-\mu}(z)\right], \qquad (2.25)
$$

$$
Q_n^{\mu}(z) = \frac{\pi}{2} \frac{\mathrm{e}^{\mathrm{i}\pi\mu}}{\sin(\pi\mu)} \left[P_n^{\mu}(z) - (-)^n P_n^{\mu}(-z)\right], \qquad (2.26)
$$

equivalence of which follows immediately from Eq. (2.9). An extension of the definition to negative integer degrees is made via the formula

$$
Q_{-n-1}^{\mu}(z) = -Q_n^{\mu}(z) + \pi \frac{\mathrm{e}^{\mathrm{i}\pi\mu}}{\sin(\pi\mu)} P_n^{\mu}(z). \qquad (2.27)
$$

One obtains

$$
Q_{-n-1}^{\mu}(z) = \frac{\pi}{2} \frac{\mathrm{e}^{\mathrm{i}\pi\mu}}{\sin(\pi\mu)} \left[P_n^{\mu}(z) + \frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu+1)} P_n^{-\mu}(z)\right] \qquad (2.28)
$$

and

$$
Q_{-n-1}^{\mu}(z) = \frac{\pi}{2} \frac{\mathrm{e}^{\mathrm{i}\pi\mu}}{\sin(\pi\mu)} \left[P_n^{\mu}(z) + (-)^n P_n^{\mu}(-z)\right]. \qquad (2.29)
$$

From Eq. (2.25) it is seen that

$$
Q_n^{-\mu}(z) = \mathrm{e}^{-\mathrm{i}2\pi\mu} \frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)} Q_n^{\mu}(z). \qquad (2.30)
$$

It should be observed that the function $Q_{-n-1}^{\mu}(z)$ does not exist if $\mu-n$ is a negative integer or zero. The functions $Q_n^{\mu}(z)$ and $Q_{-n-1}^{\mu}(z)$ (provided the latter exists, see above) are single-valued in the complex plane with a cut along the real axis from $-\infty$ to 1.

For $0 \leqslant m \leqslant n$, it is known[2] (see [18, pages 81, 82, and 85]) that

$$
Q_n^m(z) = \frac{1}{2} P_n^m(z) \ln\frac{z+1}{z-1} - W_{n-1}^m(z) \qquad (0 \leqslant m \leqslant n), \qquad (2.31)
$$

[2] In [18, pages 81, 82, and 85] expressions for $Q_n^m(z)$, equivalent to these following from our Eqs. (2.31) to (2.33), have been given in somewhat different forms (in this connection, cf footnote 1 on page 2). We have modified Robin's formulas to make them concurrent with the notation used in the present paper.

with

$$\begin{aligned}
W_{n-1}^{m}(z) &= \frac{1}{2}[\psi(n+m+1)+\psi(n-m+1)]P_{n}^{m}(z) \\
&-\frac{(-)^{m}}{2}\left(\frac{z+1}{z-1}\right)^{m/2}\sum_{k=0}^{m-1}(-)^{k}\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{z-1}{2}\right)^{k} \\
&-\frac{1}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{n}\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{z-1}{2}\right)^{k} \\
&-\frac{1}{2}\left(\frac{z^{2}-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}\frac{(k+n+m)!\psi(k+1)}{k!(k+m)!(n-m-k)!}\left(\frac{z-1}{2}\right)^{k} \qquad (0\leqslant m\leqslant n)
\end{aligned} \tag{2.32}$$

or

$$\begin{aligned}
W_{n-1}^{m}(z) &= -\frac{1}{2}[\psi(n+m+1)+\psi(n-m+1)]P_{n}^{m}(z) \\
&+\frac{(-)^{n+m}}{2}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{m-1}\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{z+1}{2}\right)^{k} \\
&+\frac{(-)^{n}}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{z+1}{z-1}\right)^{m/2}\sum_{k=0}^{n}(-)^{k}\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{z+1}{2}\right)^{k} \\
&+\frac{(-)^{n+m}}{2}\left(\frac{z^{2}-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^{k}\frac{(k+n+m)!\psi(k+1)}{k!(k+m)!(n-m-k)!}\left(\frac{z+1}{2}\right)^{k} \\
&(0\leqslant m\leqslant n).
\end{aligned} \tag{2.33}$$

Another expression for $W_{n-1}^{m}(z)$ may be deduced from the findings of Brown [31]; it is

$$\begin{aligned}
W_{n-1}^{m}(z) &= -\frac{(-)^{m}}{2}\left(\frac{z+1}{z-1}\right)^{m/2}\sum_{k=0}^{m-1}(-)^{k}\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{z-1}{2}\right)^{k} \\
&+\frac{(-)^{n+m}}{2}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{m-1}\frac{(k+n)!(m-k-1)!}{k!(n-k)!}\left(\frac{z+1}{2}\right)^{k} \\
&-\frac{1}{2}\left(\frac{z^{2}-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}\frac{(k+n+m)!\psi(k+1)}{k!(k+m)!(n-m-k)!}\left(\frac{z-1}{2}\right)^{k} \\
&+\frac{(-)^{n+m}}{2}\left(\frac{z^{2}-1}{4}\right)^{m/2}\sum_{k=0}^{n-m}(-)^{k}\frac{(k+n+m)!\psi(k+1)}{k!(k+m)!(n-m-k)!}\left(\frac{z+1}{2}\right)^{k} \\
&(0\leqslant m\leqslant n).
\end{aligned} \tag{2.34}$$

Furthermore, Snow [13, pages 55 and 56] gave a representation of $W_{n-1}^{m}(z)$ which may be easily

shown to be equivalent to

$$\begin{aligned}
W_{n-1}^{m}(z) \quad = \quad & \pm\frac{1}{2}n!(n+m)!\left(\frac{z\mp 1}{z\pm 1}\right)^{m/2}\left(\frac{z\pm 1}{2}\right)^{n}\sum_{k=0}^{n-m}\frac{1}{k!(k+m)!(n-k)!(n-m-k)!} \\
& \times[\psi(n-m-k+1)+\psi(n-k+1)-\psi(k+m+1)-\psi(k+1)]\left(\frac{z\mp 1}{z\pm 1}\right)^{k} \\
& \pm\frac{1}{2}n!(n+m)!\left(\frac{z\pm 1}{z\mp 1}\right)^{m/2}\left(\frac{z\mp 1}{2}\right)^{n} \\
& \times\sum_{k=1}^{m}(-)^{k}\frac{(k-1)!}{(k+n)!(k+n-m)!(m-k)!}\left(\frac{z\mp 1}{z\pm 1}\right)^{k} \\
& \mp\frac{(-)^{m}}{2}n!(n+m)!\left(\frac{z\pm 1}{z\mp 1}\right)^{m/2}\left(\frac{z\pm 1}{2}\right)^{n} \\
& \times\sum_{k=0}^{m-1}(-)^{k}\frac{(m-k-1)!}{k!(n-k)!(n+m-k)!}\left(\frac{z\mp 1}{z\pm 1}\right)^{k} \qquad (0\leqslant m\leqslant n).
\end{aligned} \tag{2.35}$$

Exploiting Eq. (2.31) and either of Eqs. (2.32) to (2.34), $Q_n^{-m}(z)$ may be evaluated using

$$Q_n^{-m}(z)=\frac{(n-m)!}{(n+m)!}Q_n^{m}(z) \qquad (0\leqslant m\leqslant n), \tag{2.36}$$

which is the direct consequence of Eq. (2.30). The functions $Q_{-n-1}^{\pm m}(z)$ with $0\leqslant m\leqslant n$ do not exist.

For $m>n$, it holds that [18, Eq. (63) on page 35]

$$Q_n^{m}(z)=\frac{(-)^{m}}{2}(n+m)!(m-n-1)!\left[P_n^{-m}(-z)-(-)^{n}P_n^{-m}(z)\right] \qquad (m>n). \tag{2.37}$$

The function $Q_n^{-m}(z)$ with $m>n$ does not exist.

On the cut $x\in[-1,1]$, again following Hobson [12], we define the associated Legendre function of the second kind of non-negative and negative integer degrees as

$$Q_n^{\mu}(x)=\frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\mu}\left[\mathrm{e}^{-\mathrm{i}\pi\mu/2}Q_n^{\mu}(x+\mathrm{i}0)+\mathrm{e}^{\mathrm{i}\pi\mu/2}Q_n^{\mu}(x-\mathrm{i}0)\right] \tag{2.38}$$

and

$$Q_{-n-1}^{\mu}(x)=\frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\mu}\left[\mathrm{e}^{-\mathrm{i}\pi\mu/2}Q_{-n-1}^{\mu}(x+\mathrm{i}0)+\mathrm{e}^{\mathrm{i}\pi\mu/2}Q_{-n-1}^{\mu}(x-\mathrm{i}0)\right], \tag{2.39}$$

respectively.

3 The derivatives $\partial P_n^{\mu}(z)/\partial\mu$ and $\partial P_n^{\mu}(x)/\partial\mu$

3.1 General considerations

We begin with the observation that after differentiating Eqs. (2.6) and (2.9) with respect to μ, one arrives at the following two general relations involving the derivative in question:

$$\frac{\partial P_{-n-1}^{\mu}(z)}{\partial\mu}=\frac{\partial P_n^{\mu}(z)}{\partial\mu}, \tag{3.1}$$

$$\frac{\partial P_n^{-\mu}(z)}{\partial\mu} \quad = \quad -[\psi(n+\mu+1)+\psi(n-\mu+1)]P_n^{-\mu}(z)+(-)^{n}\frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)}\frac{\partial P_n^{\mu}(-z)}{\partial\mu}. \tag{3.2}$$

Because of Eq. (3.1), in the remainder of the paper we shall focus on evaluation of $\partial P_n^\mu(z)/\partial\mu$.

We proceed to the construction of several explicit representations of $\partial P_n^\mu(z)/\partial\mu$.

From Eq. (2.5), we have the Rodrigues-type formula

$$\begin{aligned}\frac{\partial P_n^\mu(z)}{\partial\mu} &= -\frac{1}{2}P_n^\mu(z)\ln\frac{z+1}{z-1}+\psi(n-\mu+1)P_n^\mu(z)\\ &\quad+\frac{1}{2^n\Gamma(n-\mu+1)}\left(\frac{z-1}{z+1}\right)^{\mu/2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-1)^{n-\mu}(z+1)^{n+\mu}\ln\frac{z+1}{z-1}\right]. \qquad (3.3)\end{aligned}$$

This may be rewritten as

$$\frac{\partial P_n^\mu(z)}{\partial\mu}=\frac{1}{2}P_n^\mu(z)\ln\frac{z+1}{z-1}+U_n^\mu(z) \qquad (3.4)$$

with

$$\begin{aligned}U_n^\mu(z) &= -P_n^\mu(z)\ln\frac{z+1}{z-1}+\psi(n-\mu+1)P_n^\mu(z)\\ &\quad+\frac{1}{2^n\Gamma(n-\mu+1)}\left(\frac{z-1}{z+1}\right)^{\mu/2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-1)^{n-\mu}(z+1)^{n+\mu}\ln\frac{z+1}{z-1}\right]. \qquad (3.5)\end{aligned}$$

Choosing in Eq. (2.7) the upper set of signs and differentiating with respect to μ yields $\partial P_n^\mu(z)/\partial\mu$ in the form (3.4) with

$$U_n^\mu(z)=\left(\frac{z+1}{z-1}\right)^{\mu/2}\sum_{k=0}^{n}\frac{(k+n)!\psi(k-\mu+1)}{k!(n-k)!\Gamma(k-\mu+1)}\left(\frac{z-1}{2}\right)^k. \qquad (3.6)$$

An alternative representation of $U_n^\mu(z)$ is obtained if in Eq. (2.7) one chooses the lower set of signs and then differentiates with respect to μ; this results in

$$\begin{aligned}U_n^\mu(z) &= [\psi(n+\mu+1)+\psi(n-\mu+1)]P_n^\mu(z)-(-)^n\frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu+1)}\left(\frac{z+1}{z-1}\right)^{\mu/2}\\ &\quad\times\sum_{k=0}^{n}(-)^k\frac{(k+n)!\psi(k+\mu+1)}{k!(n-k)!\Gamma(k+\mu+1)}\left(\frac{z+1}{2}\right)^k. \qquad (3.7)\end{aligned}$$

Playing with Eq. (3.7) with the aid of Eq. (2.7) (with the lower signs chosen) and of the following easy to prove identities:

$$\frac{\Gamma(n+\mu+1)}{\Gamma(n'+\mu+1)}=(-)^{n+n'}\frac{\Gamma(-n'-\mu)}{\Gamma(-n-\mu)} \qquad (3.8)$$

and

$$\psi(n+\mu+1)-\psi(n'+\mu+1)=\psi(-n-\mu)-\psi(-n'-\mu) \qquad (3.9)$$

leads to

$$\begin{aligned}U_n^\mu(z) &= [\psi(n-\mu+1)+\psi(-n-\mu)]P_n^\mu(z)-\frac{1}{\Gamma(n-\mu+1)\Gamma(-n-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2}\\ &\quad\times\sum_{k=0}^{n}\frac{(k+n)!\Gamma(-k-\mu)\psi(-k-\mu)}{k!(n-k)!}\left(\frac{z+1}{2}\right)^k. \qquad (3.10)\end{aligned}$$

Furthermore, differentiating Eq. (2.8) with respect to μ gives

$$\begin{aligned}U_n^\mu(z) &= \psi(n+\mu+1)P_n^\mu(z)\pm n!\Gamma(n+\mu+1)\left(\frac{z+1}{z-1}\right)^{\mu/2}\left(\frac{z\pm1}{2}\right)^n\\ &\quad\times\sum_{k=0}^{n}\frac{\psi(k\mp\mu+1)-\psi(n-k\pm\mu+1)}{k!(n-k)!\Gamma(k\mp\mu+1)\Gamma(n-k\pm\mu+1)}\left(\frac{z\mp1}{z\pm1}\right)^k. \qquad (3.11)\end{aligned}$$

Manipulations with Eq. (3.11), similar to those which have led us from Eq. (3.7) to Eq. (3.10), give

$$
\begin{aligned}
U_n^\mu(z) &= \psi(-n-\mu)P_n^\mu(z) + \frac{n!}{\Gamma(-n-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2}\left(\frac{z+1}{2}\right)^n \\
&\quad \times \sum_{k=0}^{n}(-)^k \frac{\Gamma(k-n-\mu)}{k!(n-k)!\Gamma(k-\mu+1)}[\psi(k-\mu+1)-\psi(k-n-\mu)]\left(\frac{z-1}{z+1}\right)^k
\end{aligned}
\tag{3.12}
$$

or

$$
\begin{aligned}
U_n^\mu(z) &= \psi(-n-\mu)P_n^\mu(z) + (-)^n\frac{n!}{\Gamma(-n-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2}\left(\frac{z-1}{2}\right)^n \\
&\quad \times \sum_{k=0}^{n}(-)^k \frac{\Gamma(-k-\mu)}{k!(n-k)!\Gamma(n-k-\mu+1)}[\psi(n-k-\mu+1)-\psi(-k-\mu)]\left(\frac{z+1}{z-1}\right)^k,
\end{aligned}
\tag{3.13}
$$

according to whether the upper or the lower signs are chosen in Eq. (3.11).

Next, let us look at Eqs. (2.7) and (3.6) which give the functions $P_n^\mu(z)$ and $U_n^\mu(z)$, respectively. It is seen that both these functions are of the form $[(z+1)/(z-1)]^{\mu/2}$ times a polynomial in $z-1$ of degree n [except for the case of $\mu = m > n$, when $P_n^m(z)$ vanishes identically, see Eq. (2.10)]. Consequently, if $\mu \neq m$, it must be possible to represent $U_n^\mu(z)$ in the form of a linear combination of the functions $P_k^\mu(z)$ with degrees not exceeding n:

$$
U_n^\mu(z) = \sum_{k=0}^{n} c_{nk}^\mu P_k^\mu(z) \tag{3.14}
$$

[in the case of $\mu = m$ one may seek $U_n^\mu(z)$ in the form of an expansion analogous to (3.14) but with $P_k^\mu(z)$ replaced by $P_k^{-\mu}(-z)$; cf also Eq. (3.34)]. Comparing coefficients at $(z-1)^n[(z+1)/(z-1)]^{\mu/2}$ in Eqs. (2.7) and (3.6), we see that the coefficient c_{nn}^μ in the above expansion is

$$
c_{nn}^\mu = \psi(n-\mu+1). \tag{3.15}
$$

The reasoning leading to determination of the coefficients c_{nk}^μ with $0 \leqslant k \leqslant n-1$ is as follows.

If we differentiate the well-known Legendre identity [23, 25]

$$
\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1) - \frac{\mu^2}{1-z^2}\right] P_n^\mu(z) = 0 \tag{3.16}
$$

with respect to μ, this yields

$$
\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1) - \frac{\mu^2}{1-z^2}\right] \frac{\partial P_n^\mu(z)}{\partial \mu} = \frac{2\mu}{1-z^2}P_n^\mu(z). \tag{3.17}
$$

On exploiting the identity (3.16), we also obtain

$$
\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1) - \frac{\mu^2}{1-z^2}\right] \frac{1}{2}P_n^\mu(z)\ln\frac{z+1}{z-1} = 2\frac{\mathrm{d}P_n^\mu(z)}{\mathrm{d}z}. \tag{3.18}
$$

By subtracting Eq. (3.18) from Eq. (3.17), we find that the function $U_n^\mu(z)$ satisfies the inhomogeneous differential equation

$$
\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1) - \frac{\mu^2}{1-z^2}\right] U_n^\mu(z) = -2\frac{\mathrm{d}P_n^\mu(z)}{\mathrm{d}z} + \frac{2\mu}{1-z^2}P_n^\mu(z). \tag{3.19}
$$

For a while, let us focus on the expression on the right-hand side of Eq. (3.19). We shall show how it may be developed into a finite sum of the functions $P_k^{\mu}(z)$ with $0 \leqslant k \leqslant n-1$. To begin with, we observe that it holds that

$$\begin{aligned} -2\frac{\mathrm{d}P_n^{\mu}(z)}{\mathrm{d}z} + \frac{2\mu}{1-z^2}P_n^{\mu}(z) &= \left[(z-1)\frac{\mathrm{d}P_n^{\mu}(z)}{\mathrm{d}z} + \frac{\mu}{z+1}P_n^{\mu}(z) - nP_n^{\mu}(z)\right] \\ &\quad - \left[(z+1)\frac{\mathrm{d}P_n^{\mu}(z)}{\mathrm{d}z} + \frac{\mu}{z-1}P_n^{\mu}(z) - nP_n^{\mu}(z)\right]. \end{aligned} \tag{3.20}$$

If use is made of the well-known relation [23, 25]

$$(z^2-1)\frac{\mathrm{d}P_n^{\mu}(z)}{\mathrm{d}z} = nzP_n^{\mu}(z) - (n+\mu)P_{n-1}^{\mu}(z), \tag{3.21}$$

the expressions in the square brackets appearing on the right-hand side of Eq. (3.20) may be written as

$$(z \mp 1)\frac{\mathrm{d}P_n^{\mu}(z)}{\mathrm{d}z} + \frac{\mu}{z \pm 1}P_n^{\mu}(z) - nP_n^{\mu}(z) = \mp\frac{(n \mp \mu)P_n^{\mu}(z) \pm (n+\mu)P_{n-1}^{\mu}(z)}{z \pm 1}. \tag{3.22}$$

Consider next the recurrence relation [23, 25]

$$(2k+1)zP_k^{\mu}(z) = (k-\mu+1)P_{k+1}^{\mu}(z) + (k+\mu)P_{k-1}^{\mu}(z). \tag{3.23}$$

Evidently, it may rewritten as

$$\begin{aligned} (2k+1)(z+1)P_k^{\mu}(z) &= \left[(k-\mu+1)P_{k+1}^{\mu}(z) + (k+\mu+1)P_k^{\mu}(z)\right] \\ &\quad + \left[(k-\mu)P_k^{\mu}(z) + (k+\mu)P_{k-1}^{\mu}(z)\right]. \end{aligned} \tag{3.24}$$

Multiplying both sides of Eq. (3.24) by $(-)^{k+n}$, summing over k from $k=0$ to $k=n-1$, utilizing then the property [cf Eq. (2.6)]

$$P_0^{\mu}(z) = P_{-1}^{\mu}(z) \tag{3.25}$$

and dividing the result by $z+1$, we find

$$\sum_{k=0}^{n-1}(-)^{k+n}(2k+1)P_k^{\mu}(z) = -\frac{(n-\mu)P_n^{\mu}(z) + (n+\mu)P_{n-1}^{\mu}(z)}{z+1}. \tag{3.26}$$

If on both sides of Eq. (3.26) use is made of Eq. (2.9), the former becomes

$$\sum_{k=0}^{n-1}(2k+1)\frac{\Gamma(k+\mu+1)}{\Gamma(k-\mu+1)}P_k^{-\mu}(-z) = -\frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu)}\,\frac{P_n^{-\mu}(-z) - P_{n-1}^{-\mu}(-z)}{z+1}. \tag{3.27}$$

From this, after replacing μ with $-\mu$ and z with $-z$, we infer

$$\sum_{k=0}^{n-1}(2k+1)\frac{\Gamma(k-\mu+1)}{\Gamma(k+\mu+1)}P_k^{\mu}(z) = \frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu)}\,\frac{P_n^{\mu}(z) - P_{n-1}^{\mu}(z)}{z-1}. \tag{3.28}$$

Hence, it follows that

$$\frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu+1)}\sum_{k=0}^{n-1}(2k+1)\frac{\Gamma(k-\mu+1)}{\Gamma(k+\mu+1)}P_k^{\mu}(z) = \frac{(n+\mu)P_n^{\mu}(z) - (n+\mu)P_{n-1}^{\mu}(z)}{z-1}. \tag{3.29}$$

Combining Eqs. (3.20), (3.22), (3.28), and (3.29), we arrive at the sought expansion

$$\begin{aligned} -2\frac{\mathrm{d}P_n^{\mu}(z)}{\mathrm{d}z} &+ \frac{2\mu}{1-z^2}P_n^{\mu}(z) \\ &= \sum_{k=0}^{n-1}(-)^{k+n}(2k+1)\left[1 - (-)^{k+n}\frac{\Gamma(n+\mu+1)\Gamma(k-\mu+1)}{\Gamma(n-\mu+1)\Gamma(k+\mu+1)}\right]P_k^{\mu}(z), \end{aligned} \tag{3.30}$$

which, apart from being useful in the present context, seems to be also interesting for its own sake.

We return to the problem of determination of the coefficients c_{nk}^{μ}. In virtue of the result in Eq. (3.30), we rewrite the differential relation (3.19) as

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z}+n(n+1)-\frac{\mu^2}{1-z^2}\right]U_n^{\mu}(z)$$
$$=\sum_{k=0}^{n-1}(-)^{k+n}(2k+1)\left[1-(-)^{k+n}\frac{\Gamma(n+\mu+1)\Gamma(k-\mu+1)}{\Gamma(n-\mu+1)\Gamma(k+\mu+1)}\right]P_k^{\mu}(z). \quad (3.31)$$

On substituting the expansion (3.14) into the left-hand side of Eq. (3.31), after equating coefficients at $P_k^{\mu}(z)$ on both sides of the resulting identity, we obtain

$$c_{nk}^{\mu} = (-)^{k+n}\frac{2k+1}{(n-k)(k+n+1)}\left[1-(-)^{k+n}\frac{\Gamma(n+\mu+1)\Gamma(k-\mu+1)}{\Gamma(n-\mu+1)\Gamma(k+\mu+1)}\right]$$
$$(0 \leqslant k \leqslant n-1). \quad (3.32)$$

Plugging Eqs. (3.15) and (3.32) into Eq. (3.14), we arrive at the following representation of $U_n^{\mu}(z)$:

$$U_n^{\mu}(z) = \psi(n-\mu+1)P_n^{\mu}(z)+\sum_{k=0}^{n-1}(-)^{k+n}\frac{2k+1}{(n-k)(k+n+1)}$$
$$\times\left[1-(-)^{k+n}\frac{\Gamma(n+\mu+1)\Gamma(k-\mu+1)}{\Gamma(n-\mu+1)\Gamma(k+\mu+1)}\right]P_k^{\mu}(z). \quad (3.33)$$

With the use of the property (2.9), the above result may be transformed into

$$U_n^{\mu}(z) = \psi(n-\mu+1)P_n^{\mu}(z)$$
$$+\sum_{k=0}^{n-1}(-)^{k+n}\frac{2k+1}{(n-k)(k+n+1)}\left[P_k^{\mu}(z)-(-)^n\frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu+1)}P_k^{-\mu}(-z)\right]. \quad (3.34)$$

The last expression is of value when $0 \leqslant \mu = m \leqslant n$.

3.2 Evaluation of $[\partial P_n^{\mu}(z)/\partial\mu]_{\mu=m}$

3.2.1 The case of $0 \leqslant m \leqslant n$

For $\mu = m$, the fundamental equation (3.4) becomes

$$\left.\frac{\partial P_n^{\mu}(z)}{\partial\mu}\right|_{\mu=m} = \frac{1}{2}P_n^m(z)\ln\frac{z+1}{z-1}+U_n^m(z) \qquad (0 \leqslant m \leqslant n). \quad (3.35)$$

It follows from Eq. (3.5) that the Rodrigues-type representation of $U_n^m(z)$ is

$$U_n^m(z) = -P_n^m(z)\ln\frac{z+1}{z-1}+\psi(n-m+1)P_n^m(z)$$
$$+\frac{1}{2^n(n-m)!}\left(\frac{z-1}{z+1}\right)^{m/2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-1)^{n-m}(z+1)^{n+m}\ln\frac{z+1}{z-1}\right]$$
$$(0 \leqslant m \leqslant n). \quad (3.36)$$

If in Eq. (3.6) use is made of the well-known limiting property

$$\lim_{\mu\to m}\frac{\psi(k-\mu+1)}{\Gamma(k-\mu+1)} = (-)^{k+m}(m-k-1)! \qquad (m > k) \quad (3.37)$$

and if, whenever necessary, the gamma function is replaced by the factorial, after some straightforward manipulations we obtain the following representation of $U_n^m(z)$:

$$\begin{aligned} U_n^m(z) &= (-)^m \left(\frac{z+1}{z-1}\right)^{m/2} \sum_{k=0}^{m-1} (-)^k \frac{(k+n)!(m-k-1)!}{k!(n-k)!} \left(\frac{z-1}{2}\right)^k \\ &\quad + \left(\frac{z^2-1}{4}\right)^{m/2} \sum_{k=0}^{n-m} \frac{(k+n+m)!\psi(k+1)}{k!(k+m)!(n-m-k)!} \left(\frac{z-1}{2}\right)^k \qquad (0 \leqslant m \leqslant n). \end{aligned} \tag{3.38}$$

Similarly, from Eq. (3.7) it follows that

$$\begin{aligned} U_n^m(z) &= [\psi(n+m+1)+\psi(n-m+1)]P_n^m(z) \\ &\quad -(-)^n \frac{(n+m)!}{(n-m)!} \left(\frac{z+1}{z-1}\right)^{m/2} \sum_{k=0}^{n} (-)^k \frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!} \left(\frac{z+1}{2}\right)^k \\ &\quad (0 \leqslant m \leqslant n). \end{aligned} \tag{3.39}$$

Furthermore, from Eq. (3.11) one finds

$$\begin{aligned} U_n^m(z) &= \psi(n+m+1)P_n^m(z) + (-)^m n!(n+m)! \left(\frac{z+1}{z-1}\right)^{m/2} \left(\frac{z+1}{2}\right)^n \\ &\quad \times \sum_{k=0}^{m-1} (-)^k \frac{(m-k-1)!}{k!(n-k)!(n+m-k)!} \left(\frac{z-1}{z+1}\right)^k \\ &\quad -n!(n+m)! \left(\frac{z-1}{z+1}\right)^{m/2} \left(\frac{z+1}{2}\right)^n \\ &\quad \times \sum_{k=0}^{n-m} \frac{\psi(n-k+1)-\psi(k+1)}{k!(k+m)!(n-k)!(n-m-k)!} \left(\frac{z-1}{z+1}\right)^k \qquad (0 \leqslant m \leqslant n) \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} U_n^m(z) &= \psi(n+m+1)P_n^m(z) + n!(n+m)! \left(\frac{z-1}{z+1}\right)^{m/2} \left(\frac{z+1}{2}\right)^n \\ &\quad \times \sum_{k=1}^{m} (-)^k \frac{(k-1)!}{(k+n)!(k+n-m)!(m-k)!} \left(\frac{z+1}{z-1}\right)^k \\ &\quad +n!(n+m)! \left(\frac{z+1}{z-1}\right)^{m/2} \left(\frac{z-1}{2}\right)^n \\ &\quad \times \sum_{k=0}^{n-m} \frac{\psi(n-m-k+1)-\psi(k+m+1)}{k!(k+m)!(n-k)!(n-m-k)!} \left(\frac{z+1}{z-1}\right)^k \qquad (0 \leqslant m \leqslant n). \end{aligned} \tag{3.41}$$

Finally, from either of Eqs. (3.33) or (3.34), with the aid of Eqs. (2.13) to (2.15), one deduces that

$$\begin{aligned} U_n^m(z) &= \psi(n-m+1)P_n^m(z) - \frac{(n+m)!}{(n-m)!} \sum_{k=0}^{n-1} (-)^k \frac{2k+1}{(n-k)(k+n+1)} P_k^{-m}(-z) \\ &\quad + \sum_{k=0}^{n-m-1} (-)^{k+n+m} \frac{2k+2m+1}{(n-m-k)(k+n+m+1)} P_{k+m}^m(z) \qquad (0 \leqslant m \leqslant n) \end{aligned} \tag{3.42}$$

or equivalently

$$\begin{aligned}
U_n^m(z) \;&=\; \psi(n-m+1)P_n^m(z) - \frac{(n+m)!}{(n-m)!}\sum_{k=0}^{m-1}(-)^k\frac{2k+1}{(n-k)(k+n+1)}P_k^{-m}(-z) \\
&\quad + \sum_{k=0}^{n-m-1}(-)^{k+n+m}\frac{2k+2m+1}{(n-m-k)(k+n+m+1)} \\
&\quad \times\left[1-(-)^{k+n+m}\frac{k!(n+m)!}{(k+2m)!(n-m)!}\right]P_{k+m}^m(z) \qquad (0 \leqslant m \leqslant n). \qquad (3.43)
\end{aligned}$$

3.2.2 A closer look at the case of $m = 0$

Setting $\mu = 0$ in Eq. (3.17) results in

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1)\right]\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=0} = 0, \qquad (3.44)$$

i.e., $[\partial P_n^\mu(z)/\partial\mu]_{\mu=0}$ solves the same differential equation as the Legendre polynomial $P_n(z) \equiv P_n^0(z)$ does. Consequently, it must be possible to write it in the form of a linear combination of $P_n(z)$ and the Legendre function of the second kind $Q_n(z) \equiv Q_n^0(z)$.

To find that combination, at first we observe that for $m = 0$ Eq. (3.35) becomes

$$\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=0} = \frac{1}{2}P_n(z)\ln\frac{z+1}{z-1} + U_n^0(z). \qquad (3.45)$$

From Eq. (3.43) we see that $U_n^0(z)$ is a polynomial in z of degree n, which may be written as

$$U_n^0(z) = \psi(n+1)P_n(z) - W_{n-1}(z). \qquad (3.46)$$

Here $W_{n-1}(z)$ is a polynomial in z (of degree $n-1$) given by

$$W_{n-1}(z) = 2\sum_{k=0}^{n-1}\frac{1-(-)^{k+n}}{2}\frac{2k+1}{(n-k)(k+n+1)}P_k(z) \qquad (3.47)$$

or equivalently

$$W_{n-1}(z) = \sum_{k=0}^{\mathrm{int}[(n-1)/2]}\frac{2n-4k-1}{(n-k)(2k+1)}P_{n-2k-1}(z). \qquad (3.48)$$

A glance at Eq. (3.48) reveals that $W_{n-1}(z)$ is the well-known Christoffel's polynomial (cf, e.g., [14, page 153]), in terms of which one has [23, 25]

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\frac{z+1}{z-1} - W_{n-1}(z). \qquad (3.49)$$

On combining Eqs. (3.45), (3.46), and (3.49), we obtain the sought relationship[3] [cf Eq. (1.1)]

$$\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=0} = \psi(n+1)P_n(z) + Q_n(z). \qquad (3.50)$$

The same result may be obtained if one couples Eq. (3.2) with Eq. (4.1), both particularized to the case of $m = 0$.

[3] The counterpart relationship (see [23, page 178])

$$\left.\frac{\partial Q_n^\mu(z)}{\partial\mu}\right|_{\mu=0} = [\mathrm{i}\pi + \psi(n+1)]Q_n(z)$$

follows straightforwardly from Eq. (2.30) after differentiating the latter with respect to μ and setting then $\mu = 0$.

3.2.3 The case of $m > n$

In this case, the function $P_n^m(z)$ vanishes identically [cf Eq. (2.10)], so that Eq. (3.4) becomes

$$\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=m} = U_n^m(z) \qquad (m > n). \tag{3.51}$$

With the help of Eq. (3.37), from Eq. (3.6) we find

$$U_n^m(z) \;=\; (-)^m \left(\frac{z+1}{z-1}\right)^{m/2} \sum_{k=0}^{n} (-)^k \frac{(k+n)!(m-k-1)!}{k!(n-k)!} \left(\frac{z-1}{2}\right)^k \qquad (m > n). \tag{3.52}$$

Comparison of Eq. (3.52) with Eq. (2.20) reveals the relationship[4]

$$U_n^m(z) = (-)^m (n+m)!(m-n-1)! P_n^{-m}(-z) \qquad (m > n). \tag{3.53}$$

Consequently, from Eqs. (3.51) and (3.53) one obtains

$$\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=m} = (-)^m (n+m)!(m-n-1)! P_n^{-m}(-z) \qquad (m > n). \tag{3.54}$$

3.3 Evaluation of $[\partial P_n^\mu(z)/\partial\mu]_{\mu=-m}$

3.3.1 The case of $0 \leqslant m \leqslant n$

For $\mu = -m$, Eq. (3.4) becomes

$$\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=-m} = \frac{1}{2} P_n^{-m}(z) \ln\frac{z+1}{z-1} + U_n^{-m}(z) \qquad (0 \leqslant m \leqslant n). \tag{3.55}$$

Since it holds that

$$\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=-m} = -\left.\frac{\partial P_n^{-\mu}(z)}{\partial\mu}\right|_{\mu=m}, \tag{3.56}$$

from Eqs. (3.55), (3.2), and (3.35) it may be inferred that

$$U_n^{-m}(z) \;=\; (-)^{n+1}\frac{(n-m)!}{(n+m)!} U_n^m(-z) + [\psi(n+m+1) + \psi(n-m+1)] P_n^{-m}(z) \qquad (0 \leqslant m \leqslant n). \tag{3.57}$$

One may couple this formula with Eqs. (3.38) to (3.43) and exploit, whenever necessary, Eqs. (2.11) and (2.12) to obtain various explicit representations of $U_n^{-m}(z)$ with $0 \leqslant m \leqslant n$.

3.3.2 The case of $m > n$

From Eq. (3.4) we find

$$\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=-m} = \frac{1}{2} P_n^{-m}(z) \ln\frac{z+1}{z-1} + U_n^{-m}(z) \qquad (m > n). \tag{3.58}$$

[4] The result (3.53) may be also deduced from Eq. (3.34). Indeed, it is seen that in the case considered here the summand in the latter equation vanishes identically and only the first term on the right-hand side survives. Further, in virtue of Eqs. (2.9) and (3.37) (with k replaced by n), we have

$$\lim_{\mu\to m} \psi(n-\mu+1) P_n^\mu(z) = (-)^m (n+m)!(m-n-1)! P_n^{-m}(-z) \qquad (m > n),$$

hence, Eq. (3.53) follows. Still another way of arriving at Eq. (3.53) is to use Eq. (3.5).

Equations (3.6), (3.10), (3.12), and (3.13) supply us with the following alternative representations of $U_n^{-m}(z)$:

$$U_n^{-m}(z)=\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{n}\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{z-1}{2}\right)^{k}\qquad(m>n),\tag{3.59}$$

$$\begin{aligned}U_n^{-m}(z)&=[\psi(n+m+1)+\psi(m-n)]P_n^{-m}(z)-\frac{1}{(n+m)!(m-n-1)!}\left(\frac{z-1}{z+1}\right)^{m/2}\\&\quad\times\sum_{k=0}^{n}\frac{(k+n)!(m-k-1)!\psi(m-k)}{k!(n-k)!}\left(\frac{z+1}{2}\right)^{k}\qquad(m>n),\end{aligned}\tag{3.60}$$

$$\begin{aligned}U_n^{-m}(z)&=\psi(m-n)P_n^{-m}(z)+\frac{n!}{(m-n-1)!}\left(\frac{z-1}{z+1}\right)^{m/2}\left(\frac{z+1}{2}\right)^{n}\\&\quad\times\sum_{k=0}^{n}(-)^k\frac{(k+m-n-1)!}{k!(k+m)!(n-k)!}[\psi(k+m+1)-\psi(k+m-n)]\left(\frac{z-1}{z+1}\right)^{k}\\&\quad(m>n),\end{aligned}\tag{3.61}$$

$$\begin{aligned}U_n^{-m}(z)&=\psi(m-n)P_n^{-m}(z)+(-)^n\frac{n!}{(m-n-1)!}\left(\frac{z-1}{z+1}\right)^{m/2}\left(\frac{z-1}{2}\right)^{n}\\&\quad\times\sum_{k=0}^{n}(-)^k\frac{(m-k-1)!}{k!(n-k)!(n+m-k)!}[\psi(n+m-k+1)-\psi(m-k)]\left(\frac{z+1}{z-1}\right)^{k}\\&\quad(m>n).\end{aligned}\tag{3.62}$$

Moreover, with the aid of Eq. (2.23), from Eq. (3.33) we obtain

$$\begin{aligned}U_n^{-m}(z)&=\psi(n+m+1)P_n^{-m}(z)\\&\quad+\sum_{k=0}^{n-1}(-)^{k+n}\frac{2k+1}{(n-k)(k+n+1)}\left[1-\frac{(k+m)!(m-k-1)!}{(n+m)!(m-n-1)!}\right]P_k^{-m}(z)\\&\quad(m>n).\end{aligned}\tag{3.63}$$

3.4 The derivative $\partial P_n^{\mu}(x)/\partial\mu$

Since [cf Eq. (2.24)] the definition of $P_n^{\mu}(x)$ with $-1\leqslant x\leqslant 1$ differs from that of $P_n^{\mu}(z)$ with $z\in\mathbb{C}\setminus[-1,1]$, the derivative $\partial P_n^{\mu}(x)/\partial\mu$ requires to be considered separately.

Differentiating Eq. (2.24) with respect to μ, making use of Eq. (3.35) and exploiting the fact that

$$x+1\pm\mathrm{i}0=1+x\qquad x-1\pm\mathrm{i}0=\mathrm{e}^{\pm\mathrm{i}\pi}(1-x)\qquad(-1\leqslant x\leqslant 1)\tag{3.64}$$

results in

$$\frac{\partial P_n^{\mu}(x)}{\partial\mu}=\frac{1}{2}P_n^{\mu}(x)\ln\frac{1+x}{1-x}+U_n^{\mu}(x),\tag{3.65}$$

where, in analogy with Eq. (2.24), we define

$$U_n^{\mu}(x)=\frac{1}{2}\left[\mathrm{e}^{\mathrm{i}\pi\mu/2}U_n^{\mu}(x+\mathrm{i}0)+\mathrm{e}^{-\mathrm{i}\pi\mu/2}U_n^{\mu}(x-\mathrm{i}0)\right]=\mathrm{e}^{\pm\mathrm{i}\pi\mu/2}U_n^{\mu}(x\pm\mathrm{i}0).\tag{3.66}$$

Various representations of $U_n^{\mu}(x)$ and $U_n^{\pm m}(x)$ arise if one couples Eq. (3.66) with the formulas for $U_n^{\mu}(z)$ and $U_n^{\pm m}(z)$ derived in sections 3.1 to 3.3 (now particularized to the case of $z=x\pm\mathrm{i}0$).

4 Applications

4.1 Construction of the associated Legendre function of the second kind of integer degree and order

As the first application of the results obtained in the preceding section, below we shall construct several explicit representations of the associated Legendre function of the second kind of integer degree and order.

4.1.1 The functions $Q_n^{\pm m}(z)$ in the case of $0 \leqslant m \leqslant n$

If in Eq. (2.26) we let μ approach $\pm m$, after applying the l'Hospital rule to the right-hand side, we obtain

$$Q_n^{\pm m}(z) = \frac{1}{2}\frac{\partial P_n^{\mu}(z)}{\partial \mu}\bigg|_{\mu=\pm m} - \frac{(-)^n}{2}\frac{\partial P_n^{\mu}(-z)}{\partial \mu}\bigg|_{\mu=\pm m} \qquad (0 \leqslant m \leqslant n). \tag{4.1}$$

Combining this with Eq. (3.3) gives the following Rodrigues-type representation of $Q_n^{\pm m}(z)$ with $0 \leqslant m \leqslant n$:

$$\begin{aligned} Q_n^{\pm m}(z) \quad = \quad & -\frac{1}{2}P_n^{\pm m}(z)\ln\frac{z+1}{z-1} \\ & +\frac{1}{2^{n+1}(n\mp m)!}\left(\frac{z-1}{z+1}\right)^{m/2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-1)^{n-m}(z+1)^{n+m}\ln\frac{z+1}{z-1}\right] \\ & +\frac{1}{2^{n+1}(n\mp m)!}\left(\frac{z+1}{z-1}\right)^{m/2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-1)^{n+m}(z+1)^{n-m}\ln\frac{z+1}{z-1}\right] \\ & (0 \leqslant m \leqslant n). \end{aligned} \tag{4.2}$$

For $m=0$, Eq. (4.2) reduces to the well-known (e.g., [25, Eq. (8.836.1)]) formula

$$Q_n(z) = -\frac{1}{2}P_n(z)\ln\frac{z+1}{z-1} + \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z^2-1)^n\ln\frac{z+1}{z-1}\right]. \tag{4.3}$$

Upon making use of Eqs. (3.4) and (2.11), Eq. (4.1) implies that

$$Q_n^{\pm m}(z) = \frac{1}{2}P_n^{\pm m}(z)\ln\frac{z+1}{z-1} - W_{n-1}^{\pm m}(z) \qquad (0 \leqslant m \leqslant n), \tag{4.4}$$

with

$$W_{n-1}^{\pm m}(z) = \frac{1}{2}\left[(-)^n U_n^{\pm m}(-z) - U_n^{\pm m}(z)\right] \qquad (0 \leqslant m \leqslant n). \tag{4.5}$$

From Eq. (4.5) it is seen that

$$W_{n-1}^{\pm m}(-z) = (-)^{n+1}W_{n-1}^{\pm m}(z) \qquad (0 \leqslant m \leqslant n), \tag{4.6}$$

i.e., the parities of $W_{n-1}^{\pm m}(z)$ are opposite to those of $P_n^{\pm m}(z)$. In view of the symmetry relation

$$W_{n-1}^{-m}(z) = \frac{(n-m)!}{(n+m)!}W_{n-1}^{m}(z) \qquad (0 \leqslant m \leqslant n), \tag{4.7}$$

which is the consequence of Eqs. (4.5), (3.57), and (2.11), henceforth we shall discuss $W_{n-1}^{m}(z)$ only.

A number of explicit representations of $W_{n-1}^{m}(z)$ may be obtained if one couples Eq. (4.5) (with the plus signs chosen) with Eqs. (3.38) to (3.43). For instance, if $U_n^m(-z)$ is calculated using Eq. (3.39), while $U_n^m(z)$ is taken in the form (3.38), then Eq. (2.32) is recovered. Conversely, if $U_n^m(-z)$ is obtained from Eq. (3.38) and $U_n^m(z)$ from Eq. (3.39), this results in Eq. (2.33). If in Eq. (4.5)

both $U_n^m(-z)$ and $U_n^m(z)$ are obtained from Eq. (3.38), one recovers the Brown's formula (2.34), while if both $U_n^m(-z)$ and $U_n^m(z)$ are derived from Eq. (3.39), one arrives at

$$\begin{aligned} W_{n-1}^m(z) &= -\frac{1}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{z-1}{z+1}\right)^{m/2}\sum_{k=0}^{n}\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{z-1}{2}\right)^k \\ &\quad +\frac{(-)^n}{2}\frac{(n+m)!}{(n-m)!}\left(\frac{z+1}{z-1}\right)^{m/2}\sum_{k=0}^{n}(-)^k\frac{(k+n)!\psi(k+m+1)}{k!(k+m)!(n-k)!}\left(\frac{z+1}{2}\right)^k \\ &\quad (0 \leqslant m \leqslant n). \end{aligned} \tag{4.8}$$

That of the two Snow's representations which corresponds to the choice of the upper signs in Eq. (2.35) follows if in Eq. (4.5) $U_n^m(-z)$ is found from Eq. (3.41) and $U_n^m(z)$ from Eq. (3.40); the other one is obtained if the roles played by Eqs. (3.40) and (3.41) are interchanged.

An apparently new representation of $W_{n-1}^m(z)$ follows if in Eq. (4.5) both $U_n^m(-z)$ and $U_n^m(z)$ are found from Eq. (3.33). Proceeding in this way, with the aid of Eq. (2.11), one finds

$$\begin{aligned} W_{n-1}^m(z) &= \frac{1}{2}\frac{(n+m)!}{(n-m)!}\sum_{k=0}^{m-1}(-)^k\frac{2k+1}{(n-k)(k+n+1)}\left[P_k^{-m}(-z)-(-)^nP_k^{-m}(z)\right] \\ &\quad +\sum_{k=0}^{n-m-1}\frac{1-(-)^{k+n+m}}{2}\frac{2k+2m+1}{(n-m-k)(k+n+m+1)} \\ &\quad \times\left[1-(-)^{k+n+m}\frac{k!(n+m)!}{(k+2m)!(n-m)!}\right]P_{k+m}^m(z) \qquad (0 \leqslant m \leqslant n). \end{aligned} \tag{4.9}$$

It is seen that the summand in the second sum on the right-hand side of the above equation vanishes if $k+n+m$ is even and therefore Eq. (4.9) may be rewritten more suitably as

$$\begin{aligned} W_{n-1}^m(z) &= \frac{1}{2}\frac{(n+m)!}{(n-m)!}\sum_{k=0}^{m-1}(-)^k\frac{2k+1}{(n-k)(k+n+1)}\left[P_k^{-m}(-z)-(-)^nP_k^{-m}(z)\right] \\ &\quad +\sum_{k=0}^{n-m-1}\frac{1-(-)^{k+n+m}}{2}\frac{2k+2m+1}{(n-m-k)(k+n+m+1)} \\ &\quad \times\left[1+\frac{k!(n+m)!}{(k+2m)!(n-m)!}\right]P_{k+m}^m(z) \qquad (0 \leqslant m \leqslant n). \end{aligned} \tag{4.10}$$

Manipulating with the second sum, Eq. (4.10) may be cast into

$$\begin{aligned} W_{n-1}^m(z) &= \frac{1}{2}\frac{(n+m)!}{(n-m)!}\sum_{k=0}^{m-1}(-)^k\frac{2k+1}{(n-k)(k+n+1)}\left[P_k^{-m}(-z)-(-)^nP_k^{-m}(z)\right] \\ &\quad +\frac{1}{2}\sum_{k=0}^{\mathrm{int}[(n-m-1)/2]}\frac{2n-4k-1}{(n-k)(2k+1)} \\ &\quad \times\left[1+\frac{(n+m)!(n-m-2k-1)!}{(n-m)!(n+m-2k-1)!}\right]P_{n-2k-1}^m(z) \qquad (0 \leqslant m \leqslant n). \end{aligned} \tag{4.11}$$

It is seen that in the case of $m=0$ Eqs. (4.10) and (4.11) reduce to Eqs. (3.47) and (3.48), respectively.

4.1.2 The functions $Q_n^m(z)$ and $Q_{-n-1}^m(z)$ in the case of $m > n$

Proceeding as in section 4.1.1, from Eq. (2.26) we obtain

$$Q_n^m(z) = \frac{1}{2}\frac{\partial P_n^\mu(z)}{\partial\mu}\bigg|_{\mu=m} - \frac{(-)^n}{2}\frac{\partial P_n^\mu(-z)}{\partial\mu}\bigg|_{\mu=m} \qquad (m > n). \tag{4.12}$$

If use is made here of Eq. (3.54), this yields Eq. (2.37). Next, for the function $Q_{-n-1}^{m}(z)$, from Eq. (2.27), with the aid of the l'Hospital rule, we have

$$Q_{-n-1}^{m}(z) = -Q_{n}^{m}(z) + \left.\frac{\partial P_{n}^{\mu}(z)}{\partial \mu}\right|_{\mu=m} \qquad (m > n). \tag{4.13}$$

From this, after merging with Eqs. (2.37) and (3.54), we find

$$Q_{-n-1}^{m}(z) = \frac{(-)^{m}}{2}(n+m)!(m-n-1)!\left[P_{n}^{-m}(-z) + (-)^{n}P_{n}^{-m}(z)\right] \qquad (m > n). \tag{4.14}$$

4.1.3 The functions $Q_{n}^{\pm m}(x)$ and $Q_{-n-1}^{m}(x)$

Inserting Eq. (4.4) into Eq. (2.38), the latter being particularized to $\mu = \pm m$, with the help of Eq. (2.24) we obtain

$$Q_{n}^{\pm m}(x) = \frac{1}{2}P_{n}^{\pm m}(x)\ln\frac{1+x}{1-x} - W_{n-1}^{\pm m}(x) \qquad (0 \leqslant m \leqslant n), \tag{4.15}$$

where we define

$$W_{n-1}^{\pm m}(x) = \frac{(-)^{m}}{2}\left[\mathrm{e}^{\mp \mathrm{i}\pi m/2}W_{n-1}^{\pm m}(x+\mathrm{i}0) + \mathrm{e}^{\pm \mathrm{i}\pi m/2}W_{n-1}^{\pm m}(x-\mathrm{i}0)\right] \qquad (0 \leqslant m \leqslant n). \tag{4.16}$$

It follows from Eqs. (4.16) and (4.6) that the functions $W_{n-1}^{\pm m}(x)$ possess the reflection property

$$W_{n-1}^{\pm m}(-x) = (-)^{n+m+1}W_{n-1}^{\pm m}(x) \qquad (0 \leqslant m \leqslant n), \tag{4.17}$$

while if Eq. (4.16) is coupled with Eq. (4.7), this results in

$$W_{n-1}^{-m}(x) = (-)^{m}\frac{(n-m)!}{(n+m)!}W_{n-1}^{m}(x) \qquad (0 \leqslant m \leqslant n). \tag{4.18}$$

The latter relationship allows one to focus on $W_{n-1}^{m}(x)$ only.

Exploiting the expressions for $W_{n-1}^{m}(z)$ found in section 4.1.1, with no difficulty one may construct counterpart formulas for $W_{n-1}^{m}(x)$. For instance, if Eq. (4.16) is combined with Eq. (4.11) and use is made of Eq. (2.11), this yields

$$\begin{aligned} W_{n-1}^{m}(x) &= \frac{1}{2}\frac{(n+m)!}{(n-m)!}\sum_{k=0}^{m-1}(-)^{k}\frac{2k+1}{(n-k)(k+n+1)}\left[P_{k}^{-m}(-x) - (-)^{n+m}P_{k}^{-m}(x)\right] \\ &\quad + \frac{1}{2}\sum_{k=0}^{\mathrm{int}[(n-m-1)/2]}\frac{2n-4k-1}{(n-k)(2k+1)} \\ &\quad \times \left[1 + \frac{(n+m)!(n-m-2k-1)!}{(n-m)!(n+m-2k-1)!}\right]P_{n-2k-1}^{m}(x) \qquad (0 \leqslant m \leqslant n). \end{aligned} \tag{4.19}$$

If Eq. (2.37) is used in Eq. (2.38), one finds

$$Q_{n}^{m}(x) = \frac{(-)^{m}}{2}(n+m)!(m-n-1)!\left[P_{n}^{-m}(-x) - (-)^{n+m}P_{n}^{-m}(x)\right] \qquad (m > n), \tag{4.20}$$

while if Eq. (4.14) is coupled with Eq. (2.39), this gives

$$Q_{-n-1}^{m}(x) = \frac{(-)^{m}}{2}(n+m)!(m-n-1)!\left[P_{n}^{-m}(-x) + (-)^{n+m}P_{n}^{-m}(x)\right] \qquad (m > n). \tag{4.21}$$

4.2 Evaluation of $[\partial^2 P_n^{\mu}(z)/\partial\mu^2]_{\mu=m}$ for $m > n$

In this section, we shall show that if $m > n$, then the knowledge of $[\partial P_n^{\mu}(-z)/\partial\mu]_{\mu=-m}$ allows one to find $[\partial^2 P_n^{\mu}(z)/\partial\mu^2]_{\mu=m}$.

We begin with the observation that from the identity

$$\psi(\zeta) = \psi(1-\zeta) - \cos(\pi\zeta)\Gamma(\zeta)\Gamma(1-\zeta), \tag{4.22}$$

which may be easily deduced from elementary properties of the digamma and gamma functions [14, 20, 23, 25, 30], one finds that

$$\psi(n-\mu+1) = \psi(\mu-n) + (-)^n \cos(\pi\mu)\Gamma(\mu-n)\Gamma(n-\mu+1). \tag{4.23}$$

The identity (4.23) allows one to rewrite the relation (3.2) in the form

$$\begin{aligned} \frac{\partial P_n^{-\mu}(z)}{\partial\mu} \quad = \quad & -[\psi(n+\mu+1) + \psi(\mu-n)]P_n^{-\mu}(z) \\ & +(-)^n \frac{\Gamma(n-\mu+1)}{\Gamma(n+\mu+1)} \left[\frac{\partial P_n^{\mu}(-z)}{\partial\mu} - \cos(\pi\mu)\Gamma(\mu-n)\Gamma(n+\mu+1)P_n^{-\mu}(z)\right], \end{aligned} \tag{4.24}$$

hence, it follows that

$$\begin{aligned} \left.\frac{\partial P_n^{-\mu}(z)}{\partial\mu}\right|_{\mu=m} \quad = \quad & -[\psi(n+m+1) + \psi(m-n)]P_n^{-m}(z) \\ & + \frac{(-)^n}{(n+m)!} \lim_{\mu\to m} \left\{\Gamma(n-\mu+1) \right. \\ & \left. \times \left[\frac{\partial P_n^{\mu}(-z)}{\partial\mu} - \cos(\pi\mu)\Gamma(\mu-n)\Gamma(n+\mu+1)P_n^{-\mu}(z)\right]\right\} \\ & (m > n). \end{aligned} \tag{4.25}$$

The limit in the above equation may be evaluated with the aid of the l'Hospital rule [for this purpose, the property (3.37) proves helpful] and one finds

$$\begin{aligned} \left.\frac{\partial P_n^{-\mu}(z)}{\partial\mu}\right|_{\mu=m} \quad = \quad & -2[\psi(n+m+1) + \psi(m-n)]P_n^{-m}(z) - \left.\frac{\partial P_n^{-\mu}(z)}{\partial\mu}\right|_{\mu=m} \\ & + \frac{(-)^m}{(n+m)!(m-n-1)!} \left.\frac{\partial^2 P_n^{\mu}(-z)}{\partial\mu^2}\right|_{\mu=m} \qquad (m > n). \end{aligned} \tag{4.26}$$

If Eq. (4.26) is solved for $[\partial^2 P_n^{\mu}(-z)/\partial\mu^2]_{\mu=m}$, then use is made of Eq. (3.56) and z is replaced by $-z$, one eventually arrives at

$$\begin{aligned} \left.\frac{\partial^2 P_n^{\mu}(z)}{\partial\mu^2}\right|_{\mu=m} \quad = \quad & (-)^m 2(n+m)!(m-n-1)! \\ & \times \left\{[\psi(n+m+1) + \psi(m-n)]P_n^{-m}(-z) - \left.\frac{\partial P_n^{\mu}(-z)}{\partial\mu}\right|_{\mu=-m}\right\} \\ & (m > n). \end{aligned} \tag{4.27}$$

Several explicit representations of $[\partial^2 P_n^{\mu}(z)/\partial\mu^2]_{\mu=m}$ with $m > n$, not listed here, follow if one combines Eq. (4.27) with the results of section 3.3.2.

4.3 Evaluation of $[\partial Q_n^\mu(z)/\partial\mu]_{\mu=m}$ and $[\partial Q_{-n-1}^\mu(z)/\partial\mu]_{\mu=m}$ for $m > n$

The results of section 4.2 may be exploited to express the derivatives $[\partial Q_n^\mu(z)/\partial\mu]_{\mu=m}$ and $[\partial Q_{-n-1}^\mu(z)/\partial\mu]_{\mu=m}$ for $m > n$ in terms of $[\partial P_n^\mu(\pm z)/\partial\mu]_{\mu=-m}$.

To show this for $[\partial Q_n^\mu(z)/\partial\mu]_{\mu=m}$, we differentiate Eq. (2.26) with respect to μ, obtaining

$$\frac{\partial Q_n^\mu(z)}{\partial\mu} = \mathrm{i}\pi Q_n^\mu(z) + \frac{\pi}{\sin(\pi\mu)}\left[-\cos(\pi\mu)Q_n^\mu(z) + \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi\mu}\frac{\partial P_n^\mu(z)}{\partial\mu} - \frac{(-)^n}{2}\mathrm{e}^{\mathrm{i}\pi\mu}\frac{\partial P_n^\mu(-z)}{\partial\mu}\right]. \tag{4.28}$$

In the limit $\mu \to m > n$, with the help of the l'Hospital rule and of Eq. (4.12), from Eq. (4.28) we obtain

$$\left.\frac{\partial Q_n^\mu(z)}{\partial\mu}\right|_{\mu=m} = 2\mathrm{i}\pi Q_n^m(z) - \left.\frac{\partial Q_n^\mu(z)}{\partial\mu}\right|_{\mu=m} + \frac{1}{2}\left.\frac{\partial^2 P_n^\mu(z)}{\partial\mu^2}\right|_{\mu=m} - \frac{(-)^n}{2}\left.\frac{\partial^2 P_n^\mu(-z)}{\partial\mu^2}\right|_{\mu=m} \qquad (m > n), \tag{4.29}$$

hence, it follows that

$$\left.\frac{\partial Q_n^\mu(z)}{\partial\mu}\right|_{\mu=m} = \mathrm{i}\pi Q_n^m(z) + \frac{1}{4}\left.\frac{\partial^2 P_n^\mu(z)}{\partial\mu^2}\right|_{\mu=m} - \frac{(-)^n}{4}\left.\frac{\partial^2 P_n^\mu(-z)}{\partial\mu^2}\right|_{\mu=m} \qquad (m > n). \tag{4.30}$$

If the last two terms on the right-hand side of Eq. (4.30) are transformed with the aid of Eq. (4.27) and use is made of Eq. (2.37), this yields the sought relationship

$$\begin{aligned}\left.\frac{\partial Q_n^\mu(z)}{\partial\mu}\right|_{\mu=m} &= [\mathrm{i}\pi + \psi(n+m+1) + \psi(m-n)]Q_n^m(z) \\ &\quad -\frac{(-)^m}{2}(n+m)!(m-n-1)!\left[\left.\frac{\partial P_n^\mu(-z)}{\partial\mu}\right|_{\mu=-m} - (-)^n\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=-m}\right] \\ &\quad (m > n).\end{aligned} \tag{4.31}$$

If one starts with Eq. (2.29) rather than with Eq. (2.26), after movements almost identical to these presented above one arrives at

$$\begin{aligned}\left.\frac{\partial Q_{-n-1}^\mu(z)}{\partial\mu}\right|_{\mu=m} &= [\mathrm{i}\pi + \psi(n+m+1) + \psi(m-n)]Q_{-n-1}^m(z) \\ &\quad -\frac{(-)^m}{2}(n+m)!(m-n-1)!\left[\left.\frac{\partial P_n^\mu(-z)}{\partial\mu}\right|_{\mu=-m} + (-)^n\left.\frac{\partial P_n^\mu(z)}{\partial\mu}\right|_{\mu=-m}\right] \\ &\quad (m > n).\end{aligned} \tag{4.32}$$

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